Self-sustained trapping mechanism of zero-velocity parametric gap solitons

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(Received 18 August 1998; revised manuscript received 26 October 1998)

Under specific excitation conditions, slowly traveling parametric solitons in quadratic media with singly resonant Bragg gratings can evolve into zero-velocity localized solutions. We demonstrate numerically this phenomenon, providing physical insight in terms of momentum densities. [S1063-651X(99)11902-0]

PACS number(s): 42.65.Tg, 42.65.Ky, 42.65.Pc

A nonlinear optical response associated with feedback mechanisms supports optical bistability in a variety of material systems. In distributed feedback gratings, in particular, a linear Bragg resonance can couple with an intensity-dependent refractive index or a parametric nonlinearity to originate localized eigenstates [1–10], i.e., slowly traveling gap solitons in the resulting nonlinear photonic band gap structure (NPBS). Such solutions can be viewed as optical bits trapped within the grating for short- or long-term storage depending on their propagation speed. Although the formation of *stationary* (i.e., zero-velocity) localized states or ''still'' gap solitons in NPBS is intriguing in view of the rapidly evolving interest in transparent optical networks and all-optical memories, they have not been observed yet; their excitation remains a challenging open problem.

In this report we show that two-color parametric gap solitons (PGS) can be excited via second-harmonic (SH) generation in quadratic media with a single band gap induced by Bragg resonance with the input field at the fundamental frequency, i.e., singly resonant NPBS [8]. This addresses a self-sustained nonlinear mechanism of localizing electromagnetic energy at zero velocity in a NPBS, whose prototype is technologically available [11].

We consider a bidirectional scalar field

$$E(Z,T)=\sum_{n=1,2}\exp[-in\omega_0T]\{E_n^+(Z,T)\times\exp[i\beta_nZ-i(n-1)\Delta kZ]+E_n^-(Z,T)\exp[-i\beta_nZ+i(n-1)\Delta kZ]\}$$

propagating in the presence of a shallow corrugation of period $\Lambda=\pi/\beta_1(\omega_B)$, where the optical fundamental frequency $\omega_0=\omega_B+\Delta\omega$ is nearly resonant with the first-order Bragg frequency ω_B (i.e., $|\Delta\omega/\omega_B|\ll 1$). We drop the Bragg coupling for the optical SH, associated with the spatial SH of the grating corrugation. Besides the obvious case of a corrugation with only odd spatial harmonics (e.g., square wave gratings), this case is representative of a channel waveguide supporting fundamental and SH modes of wave vectors $\beta_n=\beta_n(n\omega_0)$, with orthogonal polarizations so as to have $\Delta k=\beta_2-2\beta_1\sim 0$ through birefringence-induced phase matching. In this case, it is well known that a grating which is made efficient for one polarization (at fundamental) bears no effect on the orthogonal polarization (SH). The parametric interaction of the four envelopes $E_{1,2}^{\pm}(Z,T)$ through a quadratic nonlinearity is ruled by the dimensionless equations [8,10]

$$i(\pm u_{1,z}^{\pm}+v_1^{-1}u_{1,t}^{\pm})+\delta_1u_1^{\pm}+u_1^{\mp}+u_2^{\pm}(u_1^{\pm})^*=0, \quad (1)$$

$$i(\pm u_{2,z}^{\pm}+v_2^{-1}u_{2,t}^{\pm})+\delta_2u_2^{\pm}+\frac{(u_1^{\pm})^2}{2}=0, \quad (2)$$

where $v_j=V_{gj}/V_{g1}$ with $j=1,2$ are the ratio of group velocities at Bragg frequency, $\delta_1=\Delta\omega/(\Gamma V_{g1})$ is the normalized frequency detuning from Bragg condition ($\Delta\omega=0$), $\delta_2=(\Delta k+2V_{g2}^{-1}\Delta\omega)/\Gamma$ is the normalized nonlinear mismatch corrected for the frequency detuning $2\Delta\omega$, and $z\equiv\Gamma Z$ and $t\equiv\Gamma V_{g1}T$ are normalized temporal and propagation coordinates, respectively, Γ being the Bragg coupling strength. The envelopes $u_j^{\pm}\equiv E_j^{\pm}/\sqrt{I_j}$, $j=1,2$ are normalized with the reference intensities $I_j=\Gamma^2/(\chi_j\chi_1)$, where χ_j are the usual nonlinear coefficients [10]. In the limit of large mismatches $|\delta_2|\gg 1$, Eqs. (1) and (2) yield equivalent cubic or Kerr nonlinearities [8,10]. In this limit PGS solutions of the bright type fill the forbidden dynamic gap $\delta_1^2+V^2<1$ (V being the normalized soliton velocity) where linear solutions are exponentially damped [10]. They exist for both positive and negative mismatches δ_2, their peak intensity being simply proportional to the absolute mismatch $|\delta_2|$. Importantly, only low-amplitude solitons such that $\delta_1\delta_2<0$ turn out to be stable, including the limit $|\delta_1|\sim 1$ for which the propagation is governed by a nonlinear Schrödinger equation [8].

To date only the excitation of a slowly traveling PGS has been addressed in singly resonant NPBS [8]. A different approach for the formation of a stationary PGS in doubly resonant NPBS, based on the merging of two coherently excited in-phase slow PGSs, has been described in Ref. [10]. Before proceeding to study how a zero-velocity PGS can be generated by means of pulsed illumination at fundamental on a singly resonant grating, it is convenient to gain insight by investigating the existence of PGS solutions. We seek traveling state solutions of Eqs. (1) of the form $u_j^{\pm}=A_j^{\pm}x_j^{\pm}(\zeta)$ with $\zeta=\gamma(z-Vt)$, $\gamma=1/\sqrt{1-V^2}$ being the Lorentz factor, and obtain the following system:

$$\pm i\dot{x}_1^{\pm} + x_1^{\mp} + \Delta_1^{\pm} x_1^{\pm} + x_2^{\pm}(x_1^{\pm})^* = 0, \tag{3}$$

$$\pm i\dot{x}_2^{\pm} + \Delta_2^{\pm} x_2^{\pm} + \chi_{\pm}\frac{(x_1^{\pm})^2}{2} = 0, \tag{4}$$

where we set $A_1^{\pm} = [(1\pm V)/(1\mp V)]^{1/4}$, $A_2^{\pm} = (A_1^{\mp})^2$, $\Delta_j = \delta_j[\gamma(1\mp V/v_j)]^{-1}$, $\chi^{\pm} = (1\pm V)[\gamma(1\mp V/v_2)(1\mp V)]^{-1}$, and the dot denotes $d/d\zeta$. The usual ''cascading'' approach [13] neglects propagation effects in Eq. (4), i.e., $\dot{x}_2^{\pm}$ is dropped. We use here a more general perturbation approach. Looking for exponentially decaying solutions at $\zeta \to \pm\infty$, we can formally solve Eq. (4) as

$$x_2^{\pm}(\zeta) = \pm i\frac{\chi^{\pm}}{2} e^{\pm i\Delta_2^{\pm}\zeta}\int_{-\infty}^{\zeta}[x_1^{\pm}(\zeta')]^2 e^{\mp i\Delta_2^{\pm}\zeta'}d\zeta'. \tag{5}$$

After repeated integration by parts in Eq. (5), we obtain

$$x_2^{\pm}(\zeta) = -\frac{\chi_{\pm}}{2\Delta_2^{\pm}}\sum_{n=0}^{+\infty}\frac{(\mp i)^n}{(\Delta_2^{\pm})^n}\frac{d^n(x_1^{\pm})^2}{d\zeta^n}. \tag{6}$$

Although the series (6) is convergent at least for $|\Delta_2^{\pm}| \geqslant 1$, in the following we are mainly interested in the first two terms. Retaining only the first term in Eq. (6), the following ansatz in terms of modulus and phase $x_1^{\pm}(\zeta) = C_{\pm}\sqrt{\sigma\eta(\zeta)}\exp[i\phi_{\pm}(\zeta)]$ [here $C_{+} = \sqrt{|\delta_2|(1-V^2)/(1+V^2)}$, $C_{-} = -\sigma C_{+}$, $\sigma \equiv \mathrm{sgn}(\delta_2)$ with the constraint $\sigma\eta \geqslant 0$] yield the Hamiltonian system

$$\dot{q} = J\nabla_q H, \tag{7}$$

where $q \equiv (\eta,\phi)^T$, $\phi \equiv \phi_{+} - \phi_{-}$, the symplectic operator $J \equiv \mathrm{diag}[-1,1]$, and the Hamiltonian

$$H = 2\eta\cos\phi + 2\delta\eta - \eta^2/2, \tag{8}$$

where the effective detuning is $\delta \equiv (\Delta_1^{+} + \Delta_1^{-})/2 = \gamma\delta_1$. A bright PGS corresponds to homoclinic orbits of the Hamiltonian (8) emanating from the origin, which can be readily obtained explicitly (Ref. [10]). These solutions exist, for a given detuning $|\delta_1| < 1$, with any absolute velocity between zero and a critical value $V \equiv V_{\mathrm{Kerr}} \equiv \sqrt{1-\delta_1^2}$, that is, they fill the entire dynamical gap $\delta_1^2 + V^2 < 1$. Now consider the effect of the additional term in the expansion (6): in this case the reduction in terms of modulus and phase variables implies an additional compatibility condition $\chi^{+}/(\Delta_2^{+})^2 = \chi^{-}/(\Delta_2^{-})^2$, which can be solved to give the following constrained discrete values of the velocity

$$V = 0, \quad V = \pm\sqrt{2v_2 - 1}. \tag{9}$$

Restricting here to the case $v_2 = V_{g2}/V_{g1} \simeq \omega_0/2\omega_0 = 0.5$, the latter condition means that only stationary solitons with $V = 0$ are described by this approach. Provided that this constraint (9) is fulfilled, q again obeys Eq. (7) with the new nonhomogeneous symplectic operator $J = J(\eta) \equiv \mathrm{diag}[-(1+\eta/\delta_2)^{-1},(1+\eta/\delta_2)^{-1}]$. Exploiting the property of invariance of the Hamiltonian fixed points with respect to the change of the symplectic [14], we conclude that at least stationary solitons still exist, their profile being a reshaping of the first-order Kerr-like solutions. In this case, however, we were not able to obtain reasonably simple analytical expressions for the solitons. It is convenient to introduce the new variable

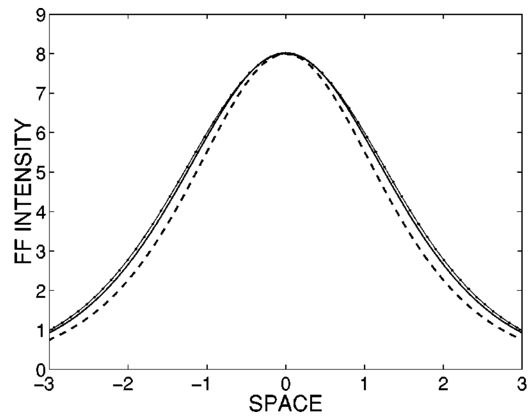

FIG. 1. Central part of the intensity profile $|u_1^{+}|^2 + |u_1^{-}|^2$ at fundamental frequency (FF) of a stationary ($V=0$) PGS, for a relatively low normalized mismatch $\delta_2 = \Delta_2^{\pm} = -5$, and $\delta_1 = \Delta_1^{\pm} = 0.8$, and $v_2 = 0.5$. We compare the first-order Kerr-like solution (dashed line), the second-order correction (solid line with dots) from Eq. (10), and the numerical solution (solid line) of Eqs. (3) and (4).

$$\zeta = \bar{\zeta} + \frac{1}{\delta_2}\int_0^{\bar{\zeta}}\eta(\zeta')d\zeta', \tag{10}$$

which permits us to reduce the new Hamiltonian system to the first-order one with $J \equiv \mathrm{diag}[-1,1]$. For a given first-order solution $\eta(\zeta)$ of Eq. (8), we obtain the second-order solution by numerically inverting Eq. (10). We point out that similar corrections for spatial parametric solitons in homogeneous (gratingless) media can be calculated explicitly [12]. The latter case, however, is much simpler because first-order solutions are real and one deals with zero-velocity solitons, the moving ones being constructed via Galileian invariance [lacking together with Lorentz invariance in Eqs. (1) and (2)].

To summarize, our second-order correction suggests that a PGS can be prolonged for relatively low mismatches Δ_2, with zero (or in general constrained to discrete values) velocity. This can give a qualitative indication that the effect of reducing the mismatch results in lower-velocity solitons. It is necessary, however, to support this heuristic argument by means of a more quantitative analysis. To do this, we seek solutions of Eqs. (3) by means of the standard numerical relaxation method. In particular, for a fixed δ_2, we seek the domain of existence in terms of velocity V of bright (we discard envelopes with a nonzero pedestal) PGS solutions of Eqs. (3) for different detuning δ_1. The profiles agree well with those found from our second-order perturbation scheme [Eq. (10)]. An example is shown in Fig. 1, where we compare the soliton profiles obtained at first order, at second order, and from the full numerical solution, respectively. The results are summarized in Fig. 2(a), where we report the critical value of velocity V below which we find PGS solutions from Eqs. (3) and (4). The curves of critical velocity are reconstructed from solutions with $\delta_1\delta_2 < 0$, sampled at circles, squares, and crosses for $\delta_2 = 2,5,20$, respectively. As

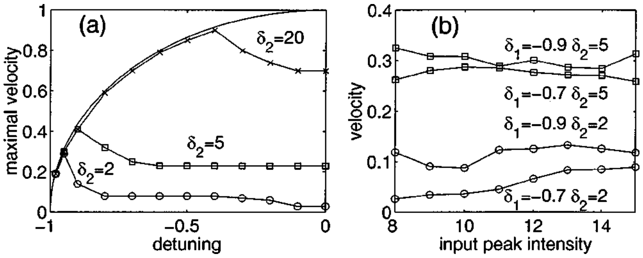

FIG. 2. (a) Critical velocity V below which soliton solutions are numerically found from Eqs. (3) and (4) versus detuning δ_1, as obtained for different mismatches δ_2. (b) Velocity V of the excited soliton versus the peak intensity P_i of an input pulse with $t_0=5$, for different values of the normalized detuning $\delta_1=-0.7$, -0.9, and mismatch $\delta_2=2$, 5.

expected, for relatively large δ_2 [crosses in Fig. 2(a)] PGSs can have relatively high velocities since the critical velocity approaches the first-order value V_{Kerr} in a large range of detunings δ_1. Clearly, as the phase mismatch (i.e., $|\delta_2|$) decreases, the critical PGS velocity decreases considerably: It approaches V_{Kerr} only for $|\delta_1|\simeq 1$, or in other words the solutions fill only a small portion of the dynamic gap. On this basis, we expect that, for a given detuning δ_1, PGSs with progressively low velocity can be excited for decreasing mismatches δ_2. It is worth pointing out that the existence of PGSs was investigated numerically for the doubly resonant case in Ref. [7], where it was shown that they do not fill the formally available existence region determined by the overlap of the two dynamical gaps at fundamental and SH, respectively. Our results, however, cannot be extrapolated from those of Ref. [7], because this existence region vanishes in the limit of negligible Bragg effect at SH.

Our aim here is to show that still PGSs can actually be launched in the NPBS. To this end, we integrated Eqs. (1) and (2) using a split-step algorithm, with a Gaussian pulse at fundamental $u_1^+(z,t)=\sqrt{P_i}\exp[-(t-z)^2/t_0^2]$, which represents illumination of a finite NPBS ($0<z<50$) from a uniform linear medium with the same average index ($z<0$). For sufficiently large input powers P_i slowly travelling PGSs are formed [8]. In Fig. 2(b) we show the results of different numerical experiments, reporting the soliton velocity V measured in the early stage of the propagation ($t<100$) against the power P_i for four different combinations of the parameter δ_1,δ_2. The results clearly indicate that the velocity is nearly independent of the incident power while it decreases for low mismatches ($\delta_2=2$), as expected. Note that the actual frequency of the formed PGS is slightly detuned with respect to δ_1 in Eqs. (1) and (2) as a consequence of the adiabatic reshaping of the fields characteristic of any soliton formation process from a nonsoliton input.

A remarkable phenomenon occurs for longer propagation times, a typical example being displayed in Fig. 3 for $\delta_1=-0.7$ and $\delta_2=2$. As shown, the fundamental input is partly reflected and partly transmitted upon generation of a SH component of a PGS inside the NPBS ($z>0$). The two-color PGS, however, reduces its propagation speed with time t, tending to the stationary state ($V=0$), a process for which we wish to coin the term ''lazy gap soliton.'' The equivalent center of mass of the PGS, in turn, moves progressively towards $z=23$. This result, rather surprising in terms of intuitive considerations on momentum conservation, can be explained on physical grounds by looking at the initial stages of the PGS formation, i.e., for $t<400$. A fraction of the generated SH freely propagates away from the input boundary ($z=0$); this characterizes the low-amplitude SH fields wherever generated by unbound fundamental components, i.e., outside the PGS. The corresponding unbound fundamentals, conversely, resonate with the Bragg grating and are subject to reflection within the NPBS, eventually counterpropagating towards the two-color gap soliton. Both these contributions tend to alter the overall momentum associated with the two-color PGS, leading to the effect pictured in Fig. 3.

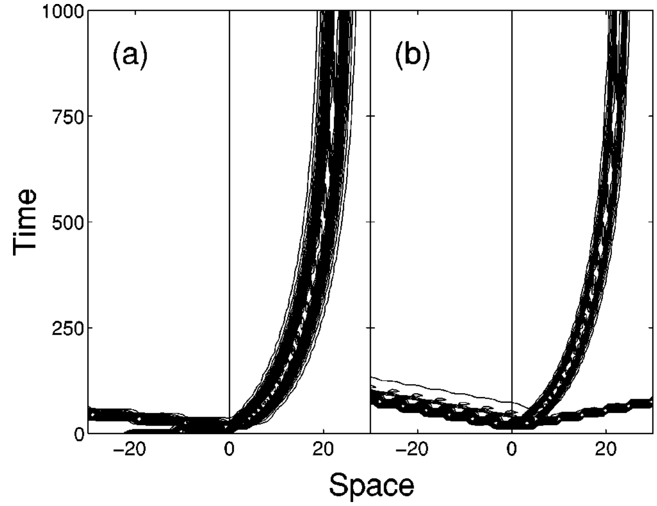

FIG. 3. Formation of a zero-velocity gap soliton in a finite Bragg grating ($0<z<50$) from a Gaussian pulse of peak intensity $P_i=12$ and width $t_0=5$ incident from a uniform medium ($z<0$): contour of the (a) fundamental; (b) SH. Here $\delta_1=-0.7$, $\delta_2=2$, $v_2=0.5$.

These considerations can be quantified in terms of momentum densities: though for a finite medium the z-translational invariance is broken and the total momentum

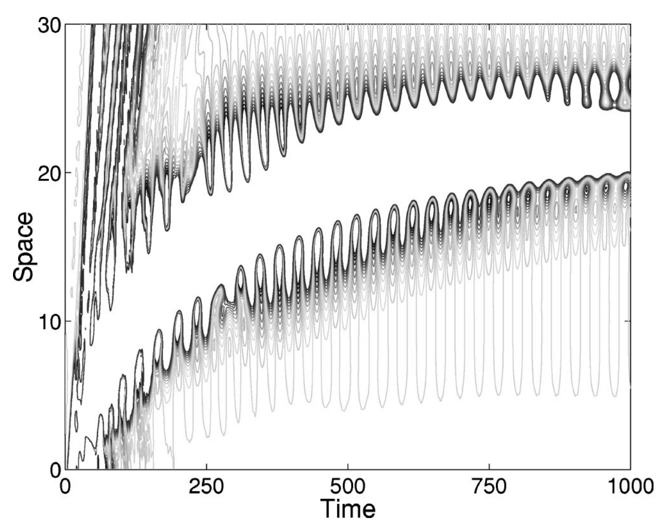

FIG. 4. Contour of the momentum density $\mathcal{M}_1$ at fundamental [Eq. (10)] versus time t and space z. The gray scale is inversely proportional to the absolute density (black curves indicate change of sign).

is not strictly conserved, we can always define a quantity $\mathcal{M}_m=\mathcal{M}_m(z,t)$ which characterizes the local density of momentum as

$$\mathcal{M}_m=\mathrm{Im}[u_m^+(\partial_z u_m^+)^*+u_m^-(\partial_z u_m^-)^*] \quad (m=1,2), \quad (11)$$

in terms of which the total momentum $M=\int_{-\infty}^{+\infty}\mathcal{M}_1+\mathcal{M}_2 dz$ is conserved for an infinite medium. We show in Fig. 4 the evolution of the density at fundamental [$m=1$ in Eq. (11)], for the case reported in Fig. 3. As shown, during the process of deceleration, the leading and trailing edges of the PGS exhibit a momentum density of different sign. The results support the heuristic arguments given above, that is, while SH momentum is lost through linear radiation, the fundamental unbound waves are Bragg reflected and contribute coherently (locally adding or subtracting) to the PGS momentum densities. This small linear wave correction supports this physical picture only in the case of low-velocity PGSs such as those investigated here.

In conclusion, a stationary PGS encompassing two (or three, in general) frequency field components, can be excited in singly resonant NPBS through the formation of ''lazy'' PGSs, i.e., slowly traveling PGS which decay to the stationary state for relatively small nonlinear phase mismatch. This phenomenon is a demonstration of the possibility of writing ''still'' optical bits through a single beam/pulse input into the structure.

This research was partially supported by the Italian MURST. The work of S.T. was carried out in the framework of an agreement with the Italian Post and Telecommunication Administration.

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